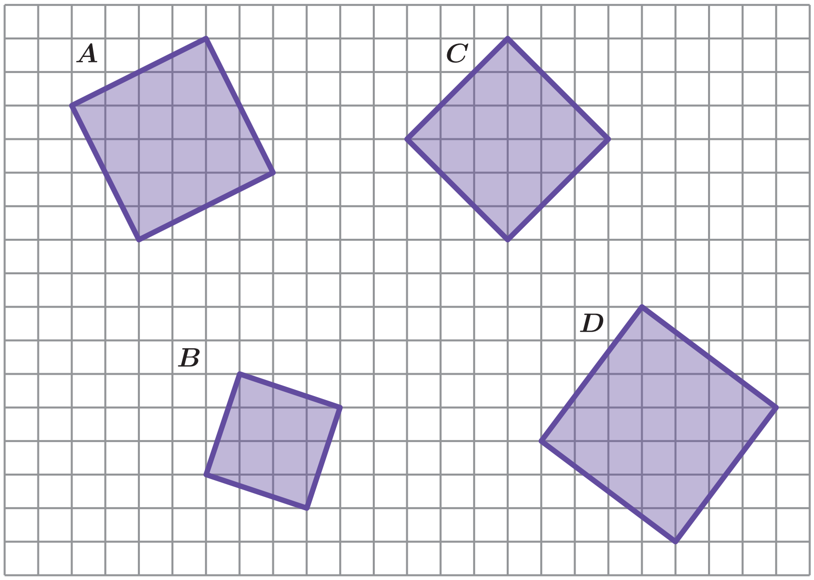
Pythagorean Theorem and Irrational Numbers

Accelerated 7
Unit 9

Synthesis

9. Describe a strategy for determining the area of a tilted square.

Use the examples if they help with your thinking.

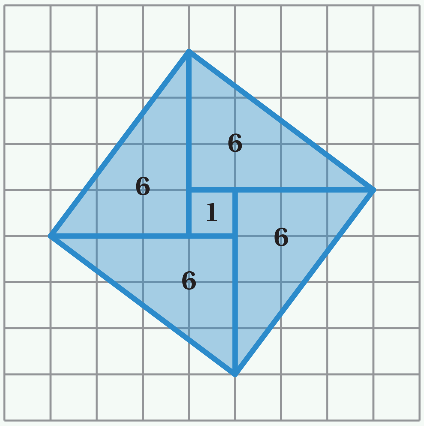


Summary

There are many strategies for determining the area of a tilted square. Here are two strategies called "decompose and rearrange" and "surround and subtract."

Decompose and Rearrange

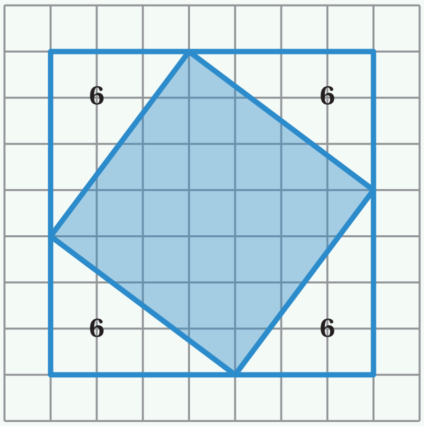
Area is 4 triangles and 1 square



$4 \cdot 6 + 1 = 25$ square units

Surround and Subtract

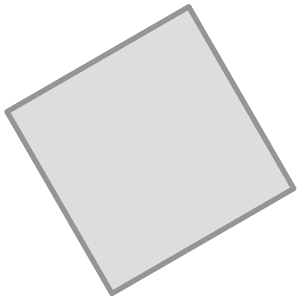
Area is the large square minus 4 triangles



$7 \cdot 7 - 4 \cdot 6 = 25$ square units

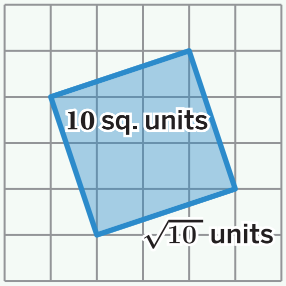
10 Synthesis

Describe the relationship between the side length and the area of a square using the term *square root*.



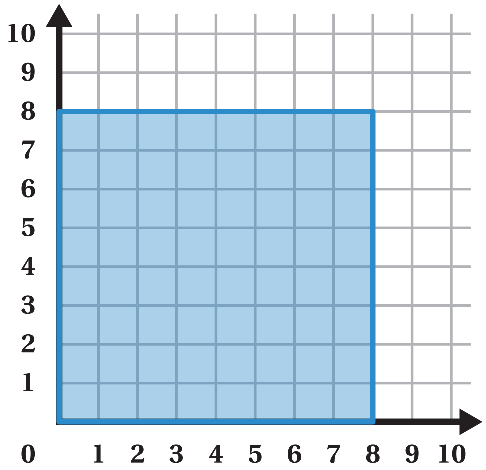
Summary

There is a known relationship between the area of any square and its side length. The *exact value* of the side length of a square can be written as the **square root** of its area. For example, $\sqrt{10}$ is the *exact value* for the side length of a square with an area of 10 square units. See the example below.



9 Synthesis

What are some strategies for approximating a square root, such as $\sqrt{75}$?



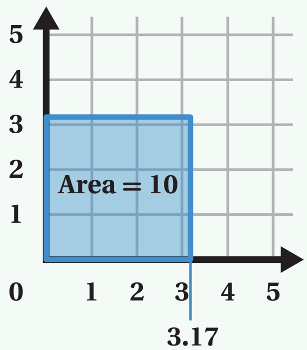
n	n^2
8.0	64

Summary

There are several strategies to approximate the values of square roots. One strategy is to use the areas of squares. Recall that the side length of a square is equal to the square root of its area. Another strategy is to create a table of values for n and determine n^2. Remember that $(\sqrt{n})^2 = n$. Below is a description of how each strategy can be used to approximate $\sqrt{10}$.

Using Squares

- Create a square that has an area equal to 10 square units.
- Approximate the side length of the square created.



Using a Table

- Create a table of decimal value guesses for n.
- Calculate n^2 for each guess of n.
- The closer n^2 is to 10, the better that value of n is as an approximation for $\sqrt{10}$.

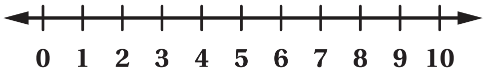
n	n^2
3.1	9.61
3.16	9.9856
3.17	10.0489
3.165	10.017225

8 Synthesis

What are some strategies for plotting square roots on a number line?

Use the number line and examples if they help to show your thinking.

$\sqrt{25}$ $\sqrt{36}$ $\sqrt{31}$ $\sqrt{40}$

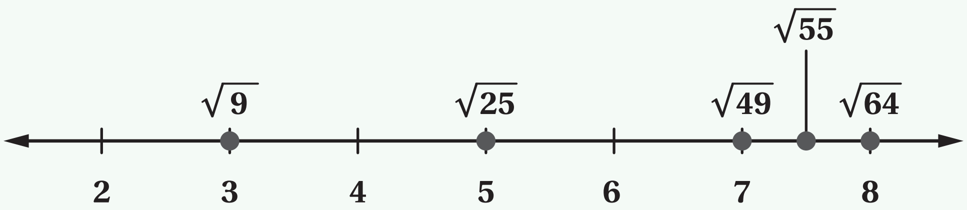


Summary

Square roots, as any number, can be represented on a number line. We write a solution to equations, such as $x^2 = 3$, using square root notation. The positive solution to this equation is $x = \sqrt{3}$.

We can approximate a square root on a number line by observing the whole numbers around it.

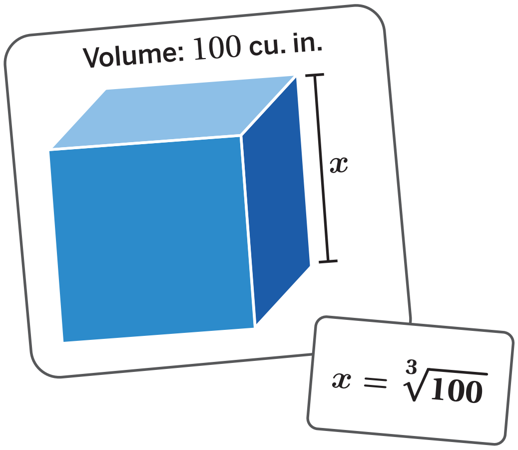
For example, you can determine that $\sqrt{55}$ is between 7 and 8 because $7^2 = 49$ and $8^2 = 64$, and 55 is between 49 and 64. More precisely, $\sqrt{55}$ should be plotted slightly left of 7.5 since it is closer to 7 than 8.

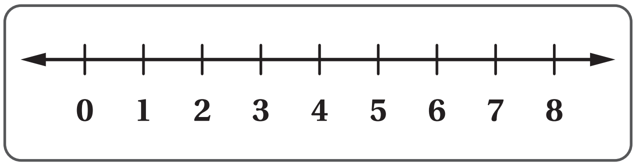


A **perfect square** is a number that is the square of an integer. For example, 49 is a perfect square because $7 \cdot 7 = 7^2 = 49$, but 55 is not a perfect square because no integer can be squared to equal 55.

13 Synthesis

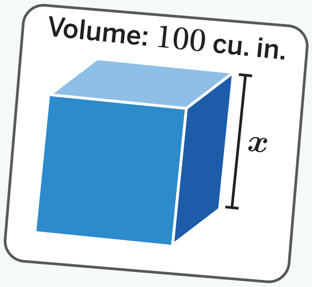
Explain a strategy for determining where to plot a cube root on the number line.



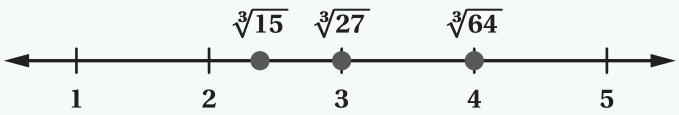


Summary

A **cube root** describes the edge length of a cube given its volume. For the cube shown with a volume of 100 cubic inches, the equation $x^3 = 100$ can help you find its edge length. Its exact solution would be represented as $x = \sqrt[3]{100}$.



We can approximate a cube root on a number line by observing the whole numbers around it. For example, you can determine that $\sqrt[3]{15}$ is between 2 and 3 because $2^3 = 8$ and $3^3 = 27$, and 15 is between 8 and 27.

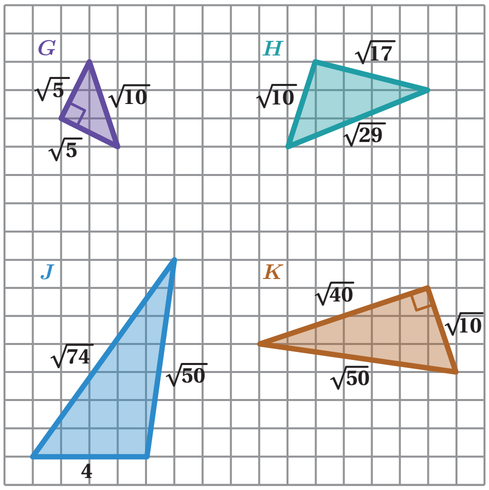


27 and 64 are *perfect cubes* because they are both the cube of an integer: $2 \cdot 2 \cdot 2 = 2^3 = 8$ and $3 \cdot 3 \cdot 3 = 3^3 = 27$.

Synthesis

7. The **Pythagorean theorem** says that for right triangles, $a^2 + b^2 = c^2$. The date of the first discovery is unknown, but the Babylonians used the Pythagorean theorem over 3,500 years ago (1,000 years before Pythagoras was born).

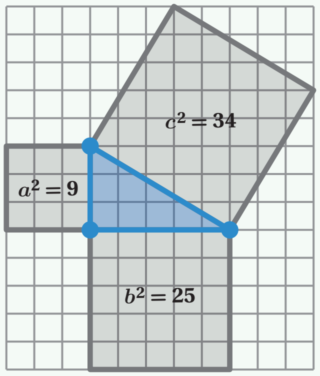
Explain the Pythagorean theorem in your own words. Use the triangles if they help with your thinking.



Summary

The **Pythagorean theorem** says that for right triangles, $a^2 + b^2 = c^2$, where a and b represent the lengths of the two shorter sides and c represents the length of the longest side.

The relationship $a^2 + b^2 = c^2$ is only true for right triangles.



For triangle H:

$(\sqrt{10})^2 + (\sqrt{17})^2 = 27$

$c^2 = (\sqrt{29})^2 = 29$

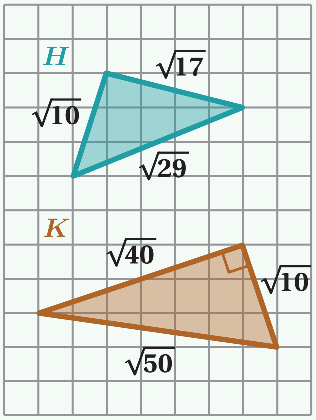
$27 \neq 29$ so $a^2 + b^2 = c^2$ is not true.

For triangle K:

$(\sqrt{10})^2 + (\sqrt{40})^2 = 50$

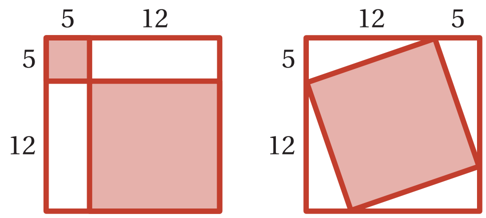
$c^2 = (\sqrt{50})^2 = 50$

$50 = 50$ so $a^2 + b^2 = c^2$ is true.



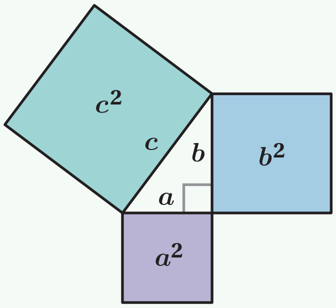
10 Synthesis

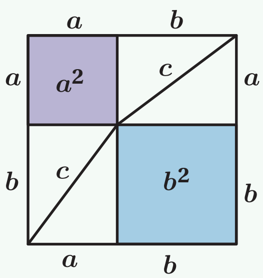
Explain how the equation $5^2 + 12^2 = 13^2$ is related to the figures on the right and to the Pythagorean theorem.

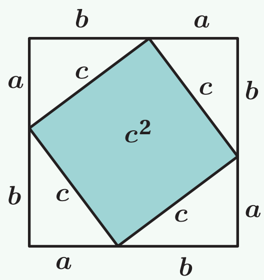


Summary

There are many proofs for the Pythagorean theorem. For example, by rearranging right triangles into squares, you saw why $a^2 + b^2$ as represented by areas of two squares, built on the legs is equal to c^2, as represented by the area of a square built on the hypotenuse.



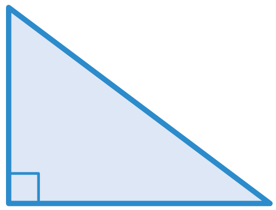




8 Synthesis

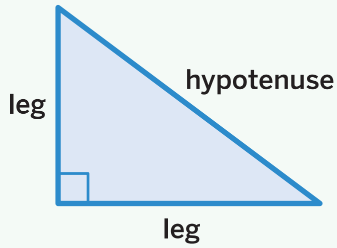
If you know two side lengths of a right triangle, how can you calculate the third side length?

Use the image if it helps to show your thinking.



Summary

The **hypotenuse** is the side of a right triangle that is opposite the right angle, and the longest side. Only right triangles have a *hypotenuse*. The **legs** of a right triangle are the sides that make the right angle.

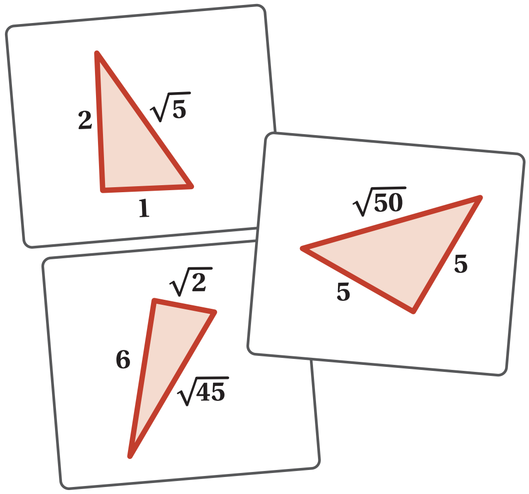


The Pythagorean theorem says that in a right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse. This can be represented by the equation $a^2 + b^2 = c^2$ where a and b represent the lengths of the legs, and c represents the length of the hypotenuse.

When any two side lengths of a right triangle are known, the Pythagorean theorem can be used to calculate the length of the third side, whether it is the hypotenuse or a leg. You can substitute the lengths you know into the equation $\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2$ or $a^2 + b^2 = c^2$, and then solve for the unknown value.

10 Synthesis

How can you tell from just the side lengths if a triangle is a right triangle?

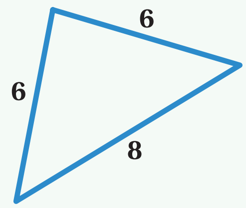


Summary

Mathematicians sometimes think about a statement's *converse*, which is a statement in the opposite direction.

If a triangle has side lengths a, b, and c, where c is the longest side and $a^2 + b^2 = c^2$, then the converse of the Pythagorean theorem says that you must have a right triangle. We can use the converse of the Pythagorean theorem to determine whether any triangle is a right triangle. If the sides of a triangle *do not* make the equation $a^2 + b^2 = c^2$ true, then you know it is *not* a right triangle.

In the triangle shown, let $a = 6$, $b = 6$, and $c = 8$. You can use substitution to determine whether the triangle is a right triangle.

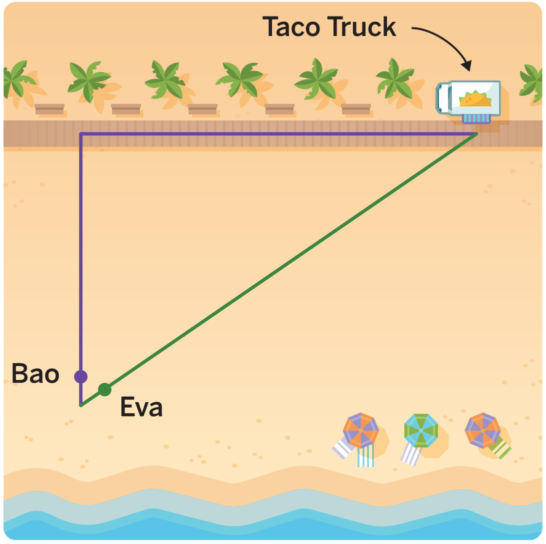


$$a^2 + b^2 = 36 + 36$$
$$= 72$$

Because $c^2 = 8^2$, or 64, the triangle cannot be a right triangle because $a^2 + b^2 \neq c^2$.

10 Synthesis

What are some important things to remember when using the Pythagorean theorem to solve problems?



Summary

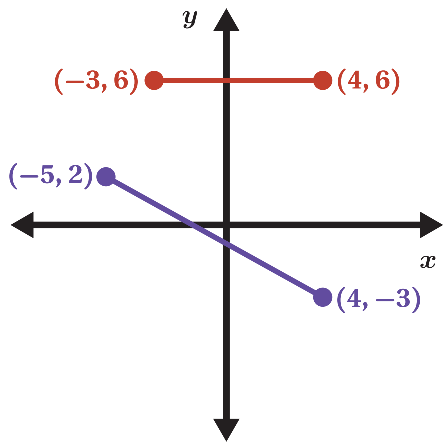
The Pythagorean theorem can be used to solve problems that can be modeled with right triangles. The sides of a triangle might represent units such as the length of an object or the distance between two objects.

To apply the Pythagorean theorem, the lengths of two sides must be known so the length of the third side can be determined.

7 Synthesis

What are some strategies to calculate the distance between two points on the coordinate plane?

Use the examples in the graph if they help to show your thinking.

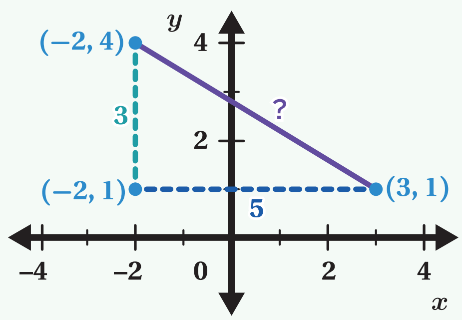


Summary

You can use the Pythagorean theorem to calculate the distance between two points that are on a diagonal line segment. To do this, start by drawing horizontal and vertical legs to form a right triangle. Then use the Pythagorean theorem to calculate the length of the hypotenuse, which will be the distance between the two points.

When two points are on a horizontal line segment, you can calculate the distance between them by determining the absolute value of the difference between their x-coordinates. For the points $(-2, 1)$ and $(3, 1)$ the distance is $|-2 - 3| = 5$ units.

Similarly, when two points are on a vertical line segment, you can calculate the distance between them by determining the absolute value of the distance between their y-coordinates. For the points $(-2, 4)$ and $(-2, 1)$ the distance is $|4 - 1| = 3$ units.



leg² + leg² = hypotenuse²

$$3^2 + 5^2 = x^2$$

$$9 + 25 = x^2$$

$$34 = x^2$$

$$\sqrt{34} = x$$

Synthesis

What is important to remember when writing a fraction as a decimal? Include how to show which digit(s) of the decimal repeat.

Use the example if it helps you with your explanation.

$$
\begin{array}{r}
0.833 \\
6\overline{)5.000} \\
-0 \\
\hline
50 \\
-48 \\
\hline
20 \\
-18 \\
\hline
20 \\
-18 \\
\hline
2
\end{array}
$$

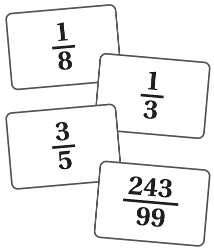
Summary

Long division can be used to represent fractions as decimals. Sometimes, a decimal is a **terminating decimal**, which means that it ends. Other times, a decimal is a **repeating decimal**, where one or more of its digits (not all zeros) repeats forever. A repeating decimal can be written with a bar over the digits which repeat or with the ellipses (. . .) at the end.

Examples		
$\frac{1}{3} = 0.333\ldots = 0.\overline{3}$	$\frac{14}{99} = 0.14141414\ldots = 0.\overline{14}$	$\frac{53}{90} = 0.5888\ldots = 0.5\overline{8}$ Remember to put the bar *only* over the repeating digit(s).

Synthesis

10. Explain a strategy for writing fractions as decimals.



Summary

Every number can be written as a decimal. Some fractions can be written as terminating decimals, while others can be written as repeating decimals. To write a fraction as a decimal, you can use long division.

For example, here is how you can use long division to rewrite $\frac{1}{15}$ as a decimal.

$$\begin{array}{r} 0.06666... \\ 15\overline{)1.00} \\ -90 \\ \hline 100 \\ -90 \\ \hline 100 \\ -90 \\ \hline 100 \\ -90 \\ \hline 10 \end{array}$$

To avoid writing the repeating part of a decimal over and over, you can use **bar notation**, which shows a line over the part of the decimal that repeats. For example, when writing $\frac{1}{15}$ as a decimal, you would write $0.06666...$ as $0.0\overline{6}$.

11 Synthesis

How is writing a repeating decimal as a fraction like writing a terminating decimal as a fraction? How is it different?

$0.3\overline{1}$ $0.8\overline{3}$ 1.25 0.4

Summary

All repeating decimals can be expressed as fractions. One way to do this is to multiply equations by factors of 10 until the repeating decimals can subtract to 0. Once the repetition is removed, the resulting equation can be solved and left in fraction form.

For example, see these steps to represent $0.\overline{57} = 0.575757575...$ as a fraction.

$$x = 0.\overline{57}$$
$$10x = 5.\overline{75}$$
$$100x = 57.\overline{57}$$

$$\begin{aligned} 100x &= 57.\overline{57} \\ -(x &= 0.\overline{57}) \\ \hline 99x &= 57 \\ x &= \frac{57}{99} \end{aligned}$$

$$0.\overline{57} = \frac{57}{99}$$

If a decimal expansion of a number is a repeating or terminating decimal, the number can be written as a fraction. If the digits in the decimal expansion do not repeat (non-repeating) and do not terminate (non-terminating), then the number cannot be written as a fraction.

9 Synthesis

What is an irrational number? Give at least one example.

Summary

A **rational number** is a number that can be written as a fraction of two non-zero integers. An **irrational number** is a number that cannot be written as a fraction of two non-zero integers.

Here are some examples of rational and irrational numbers.

Examples of Rational Numbers	Examples of Irrational Numbers
• Fractions: $\frac{10}{5}$, $3\frac{11}{20} = \frac{71}{20}$ • Terminating decimals: $1.5 = \frac{3}{2}$, $1.73 = \frac{173}{100}$ • Repeating decimals: $1.\overline{73} = \frac{172}{99}$, $0.1212\ldots = \frac{12}{99}$ • Square roots of perfect squares and cube roots of perfect cubes: $\sqrt[3]{8} = \frac{2}{1}$, $\sqrt{64} = \frac{8}{1}$, $\sqrt{\frac{1}{9}} = \frac{1}{3}$	• Non-terminating, non-repeating decimals: π, $0.743\ldots$, $2.742050\ldots$ • Square roots of non-perfect squares and cube roots of non-perfect cubes: $\sqrt{2}$, $3 \cdot \sqrt{5}$, $\sqrt[3]{9}$