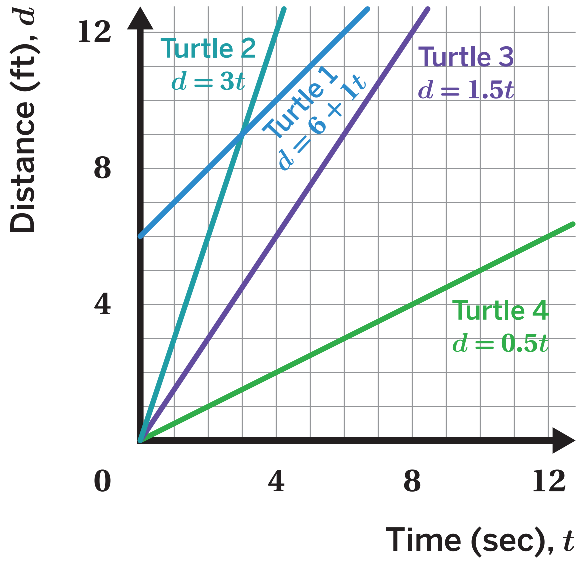
Linear Relationships and Systems of Linear Equations

Accelerated 7
Unit 4

Synthesis

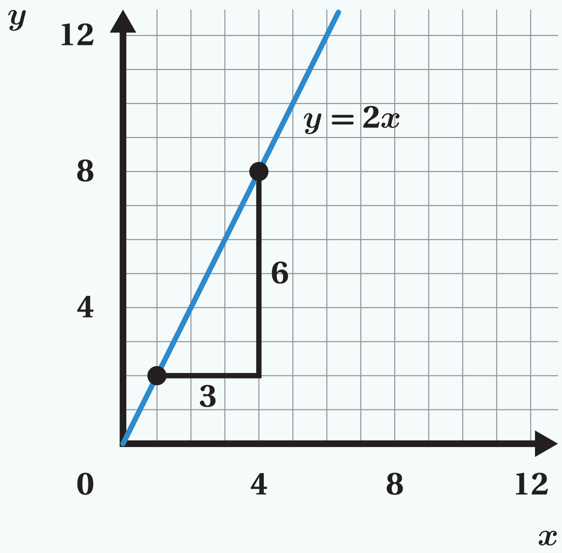
How can you tell if a relationship is proportional based on its graph or equation?



Summary

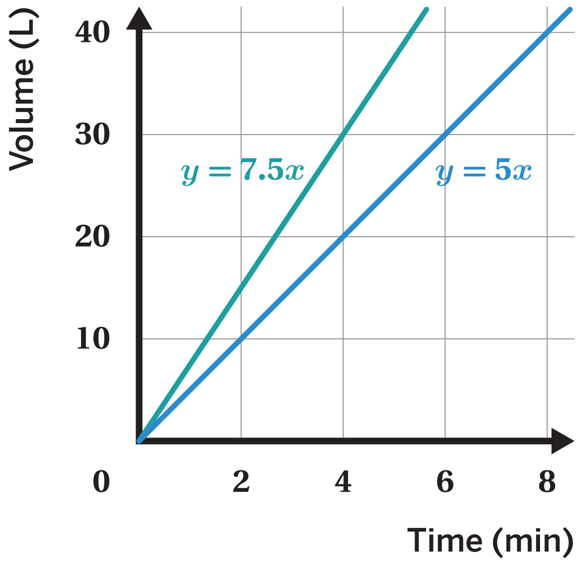
Graphs of proportional relationships are lines that pass through the origin, $(0, 0)$. For proportional relationships, the slope of the line has the same value as the unit rate.

Here is a graph of the equation $y = 2x$. The slope of the line is $\frac{6}{3}$, or 2.



Synthesis

How is the equation of a proportional relationship related to its graph?

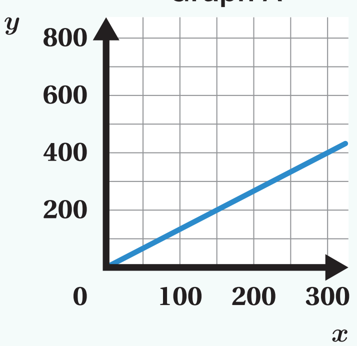


Summary

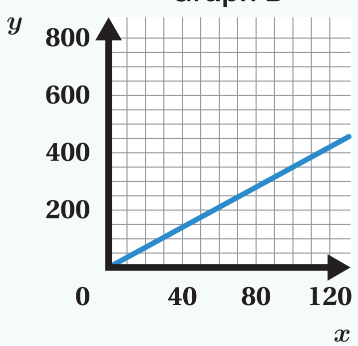
Proportional relationships can be represented by the equation $y = kx$, where k is the constant of proportionality. The constant of proportionality has the same value as the **slope**, which represents the amount y changes when x increases by 1. The slope is sometimes called a *rate of change*. Therefore, the equation $y = mx$ also represents a proportional relationship where m is the slope of the line.

The scale of the axes influences the appearance of a graph. For example, here are three graphs with differently scaled axes.

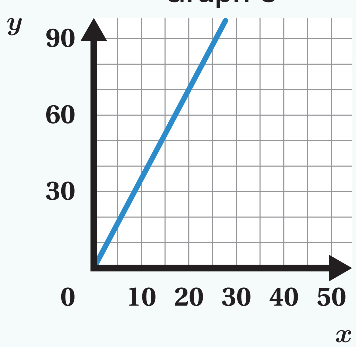
Graph A



Graph B



Graph C



Visually, it appears that the slopes of the lines in Graphs A and B are the same. However, by finding the unit rate using a point from each line, it can be determined that the slope of the lines of Graphs B and C are actually equivalent, and less than the slope of the line of Graph A.

Synthesis

1. What information do you need in order to compare two proportional relationships?

2. How did you decide which representation to use to solve the different types of problems?

Summary

Using different representations, including visual displays, can help compare proportional relationships.

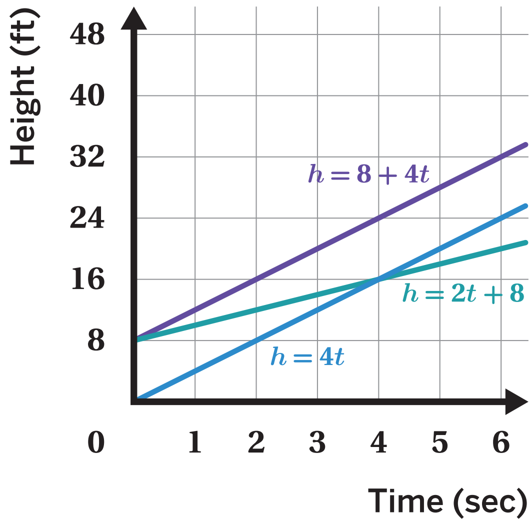
When given more than one proportional relationship — even if they are represented differently — the slope (or the unit rate) can be found from each representation and used to compare the relationships. By creating visual representations of proportional relationships, it helps to communicate thinking more clearly. This also helps others interpret numerical data from each relationship in a way that is easier to understand.

Synthesis

How can you tell if a linear relationship is non-proportional . . .

1. From a graph?

2. From an equation?



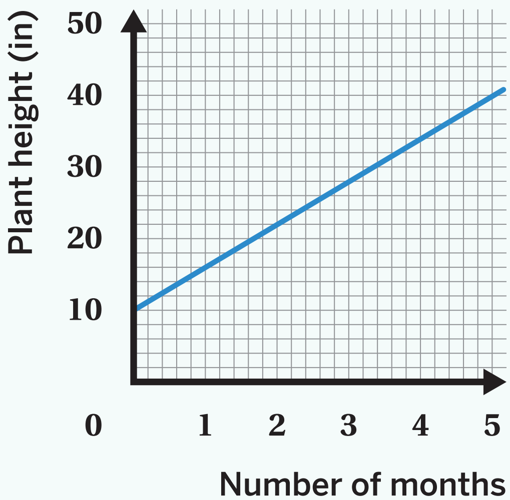
Summary

A proportional relationship is a special type of **linear relationship**, but not all linear relationships are proportional.

For example, the graph displays the height, in inches, that an outdoor plant has grown per month since it was first purchased.

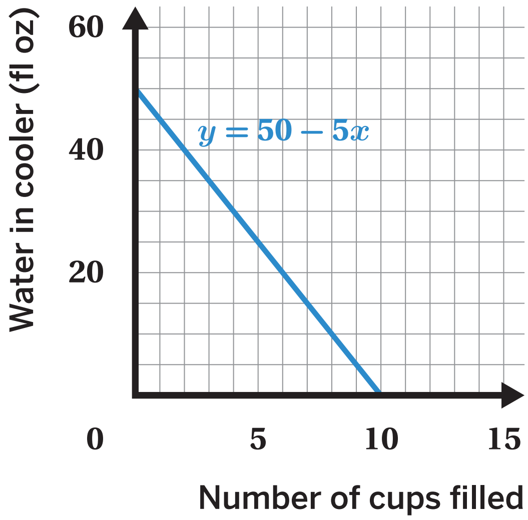
- The line starts at $(0, 10)$, which indicates the height of the plant when it was purchased was 10 inches.
- The slope is 6, which represents the number of inches the plant grew each month. The equation $y = 10 + 6x$ represents the height of the plant y after x months.

While the relationship shown is linear, it is non-proportional because the line does not pass through the origin.



Synthesis

Write 2–3 things you can determine about a situation from its graph. Use the graph if it helps your thinking.

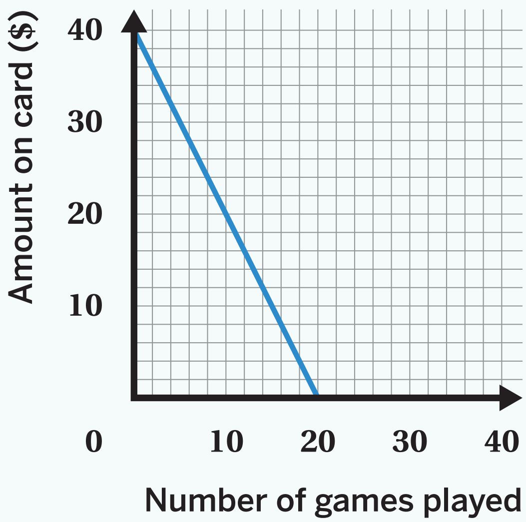


Summary

The slope of a line can have a negative value. When a linear relationship has a negative slope, this means that as the x-values increase, the y-values decrease at a constant rate.

For example, the equation $y = 40 - 2x$ represents the amount of money, y, remaining on a game card after x games have been played.

- The **vertical intercept**, also called the y-*intercept*, is $(0, 40)$ and represents the initial amount of money on the game card.
- The slope is -2 and represents the rate of change in the amount of money each time a game is played. Because the slope is negative, the amount of money decreases.
- The **horizontal intercept**, also called the x-*intercept*, is $(20, 0)$ and represents how many games can be played before the game card runs out of money.



Synthesis

Choose one representation and discuss with a partner how you know whether the slope of a linear relationship will be positive or negative.

- Graph
- Equation
- Description

Summary

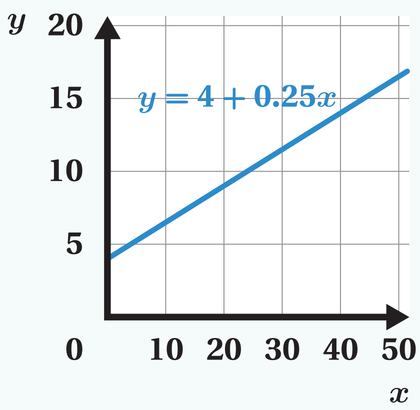
Linear equations can be written in the form $y = mx + b$ where m represents the slope and b represents the vertical intercept.

For linear relationships with a positive slope, as the x-values increase, the y-values increase at a constant rate. For linear relationships with a negative slope, as the x-values increase, the y-values decrease at a constant rate.

Here are two examples.

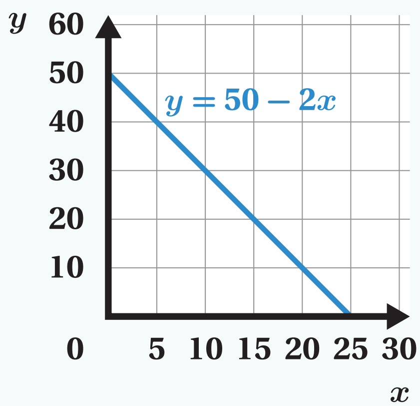
Positive slope: A medium size frozen yogurt costs $\$4$, plus $\$0.25$ per topping.

Let y represent the total cost of the frozen yogurt after adding x toppings.



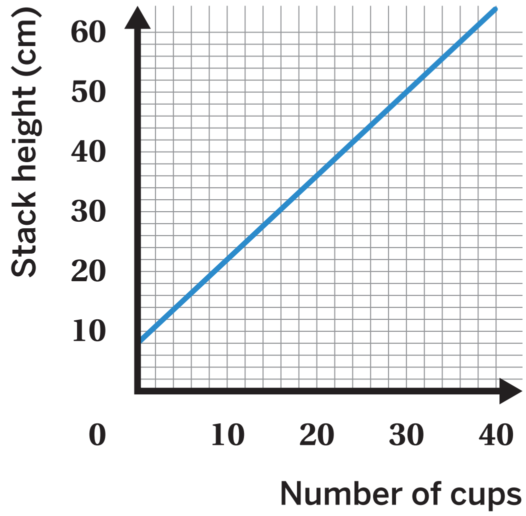
Negative slope: A student loads an arcade game card with $\$50$. Every time she plays a game, $\$2$ is subtracted from the amount available on the game card.

Let y represent the amount in dollars on the card after the student plays x games.



Synthesis

How can you use a linear relationship to make a prediction, such as determining the number of stacked cups needed to reach the top of a table?



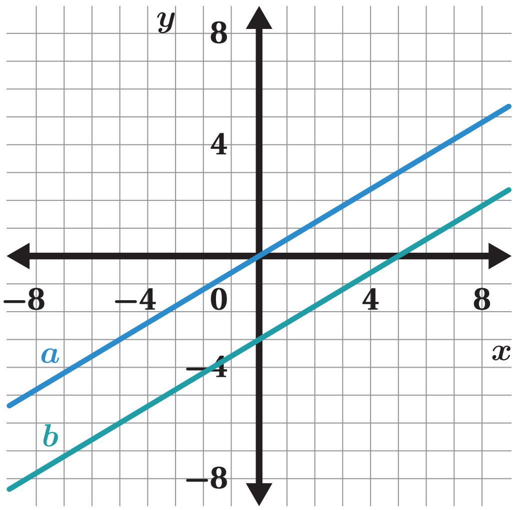
Summary

Linear relationships can be used to make predictions. You can use the graph or the equation of a linear relationship to determine the value of one variable when given the other.

For example, if you want to know how many stacked cups will have a height of 50 centimeters, you can look at a graph (like the one in the Synthesis) and determine the number of cups, x, that correspond to $y = 50$. In this context, the slope of the line tells us how much the height of the stacked cups increases every time a cup is added to the stack. You might try to intepret the y-intercept as the stack height when there are 0 cups, but this does not make sense in context. Instead, the y-value tells us the distance from the bottom of the first cup to its rim. Unlike a proportional relationship, the graph of this linear relationship does not pass through the origin.

Synthesis

Here is a graph of two equations.



1. How is the equation of line b similar to the equation of line a?

2. How is the equation of line b different from the equation of line a?

Summary

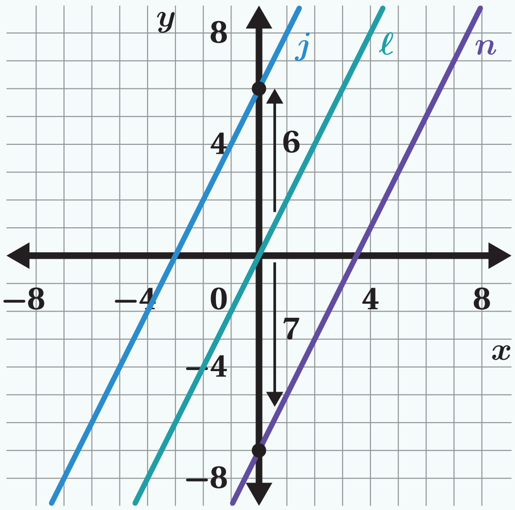
A translation of a line that represents a proportional relationship creates a line that is parallel to the pre-image, but changes the location of the vertical intercept.

The equation $y = mx$ represents a line that passes through the origin. The equation $y = mx + b$ represents a vertical translation of line $y = mx$ by b units.

- If $b > 0$, the line is translated up.
- If $b < 0$, the line is translated down.

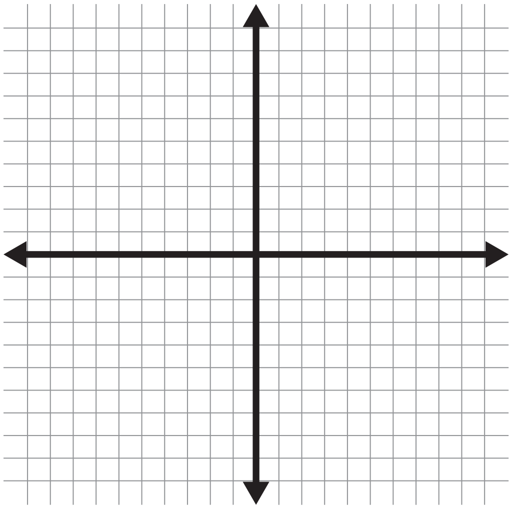
For example, the equation of line ℓ is $y = 2x$.

- Line ℓ is translated 6 units up to produce line j. So, the equation of line j is $y = 2x + 6$.
- Line ℓ is translated 7 units down to produce line n. So, the equation of line n is $y = 2x - 7$.



Synthesis

What are some strategies for finding the slope of a line that passes through two given points? Use the graph if it helps to show your thinking.



Summary

The slope of a line can be determined by two points on the line. Lines with positive slopes increase in height from left to right, while lines with negative slopes decrease in height from left to right. Slope triangles can be used to calculate the vertical and horizontal change between two points on a coordinate plane. The slope can also be calculated by listing the coordinates in a table and determining the difference between the y-coordinates — the vertical change — and the difference between the x-coordinates — the horizontal change. The slope is the ratio of the vertical change to the horizontal change.

Synthesis

How can you tell from looking at a linear equation that its graph is:

- A horizontal line?
- A vertical line?

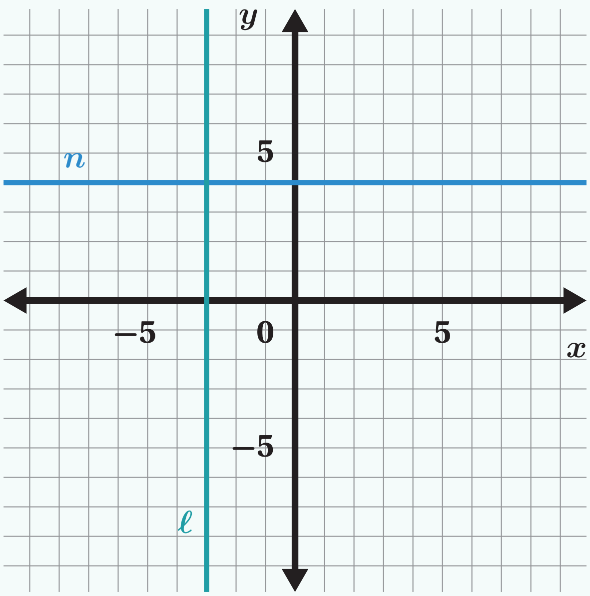
Summary

On the coordinate plane . . .

- Horizontal lines represent situations where the y-value is constant and the x-values change. Horizontal lines have a slope of 0.
- Vertical lines represent situations where the x-value is constant and the y-values change. Vertical lines have an undefined slope.

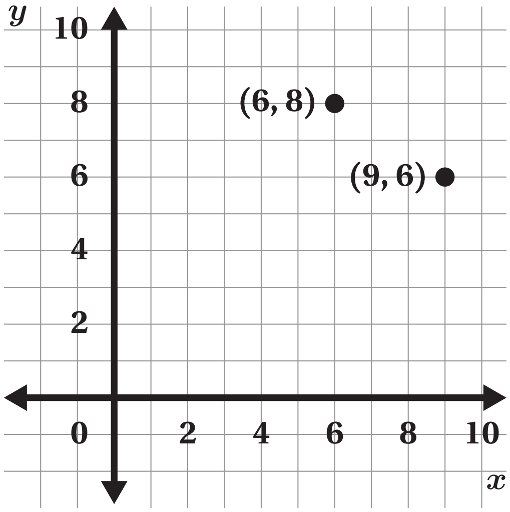
For example, the horizontal line n shown is represented by the equation $y = 4$ because every point on the horizontal line has the same y-coordinate, 4.

The vertical line ℓ shown is represented by the equation $x = -3$ because every point on the vertical line has the same x-coordinate, -3.



Synthesis

Describe how to write an equation of a line that passes through two points. Use the example if it helps with your thinking.



Summary

The equation of a line can be determined using two points on that line. Here are two different strategies that can be used to write the equation of a line using two given points.

Strategy 1:

First calculate the slope. Then substitute the coordinates of one of the points into the equation $y = mx + b$ to determine the y-intercept. Lastly, write the equation in the form $y = mx + b$.

	x	y	
+2	1	8	−6
	3	2	

slope: $\frac{-6}{2} = -3$

$y = -3x + b$

Substitute (1, 8) in for x and y.

$8 = -3(1) + b$

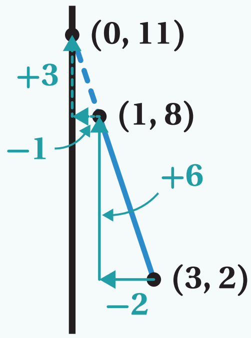
$8 = -3 + b$

$11 = b$

$y = -3x + 11$

Strategy 2:

Draw a line and use similar triangles to determine the slope and y-intercept of the line.



$8 + 3 = 11$

y-intercept: $(0, 11)$

Slope: -3

$y = -3x + 11$

Synthesis

Discuss How can you determine whether an ordered pair is a solution to a linear equation using . . .

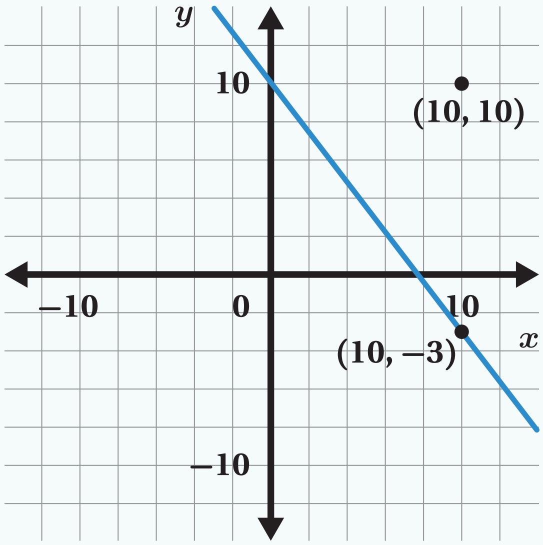
- The equation?
- A graph of the equation?

Summary

A solution to an equation with two variables is a set of values that makes the equation true. It can be expressed as an ordered pair (x, y). Every point that lies on the line is a solution to the equation. Points that do not lie on the line are not solutions to the equation.

For example, consider the linear equation $3x + 2y = 24$.

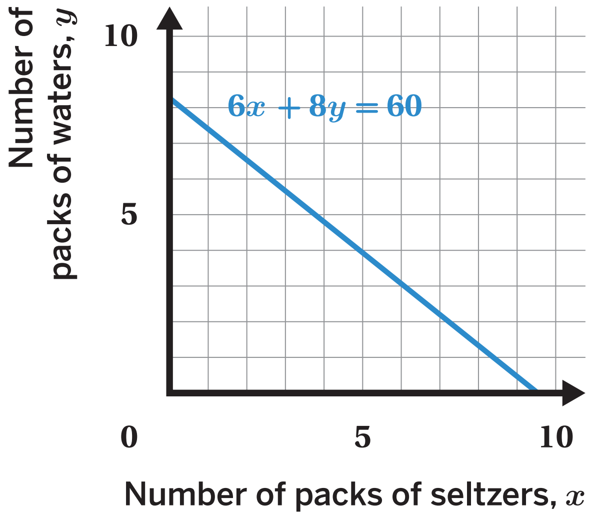
- The point $(10, -3)$ is on the line $3x + 2y = 24$. The ordered pair $(10, -3)$ is a solution to the equation $3x + 2y = 24$ because it makes the equation true; $3(10) + 2(-3) = 24$.
- The point $(10, 10)$ is not on the line. The ordered pair $(10, 10)$ is not a solution because $3(10) + 2(10) = 50$, not 24.
- Although we cannot see it on the graph, $(-10, 27)$ is also a solution to the equation $3x + 2y = 24$ because it makes the equation true; $3(-10) + 2(27) = 24$.



Synthesis

Sydney's aunt plans to purchase 60 beverages for a picnic. Seltzers are sold in packages of 6. Waters are sold in packages of 8. Here is a graph and an equation that represents the situation.

Describe how you could use the graph or equation to determine whether Sydney should buy 6 packs of seltzer and 2 packs of water.



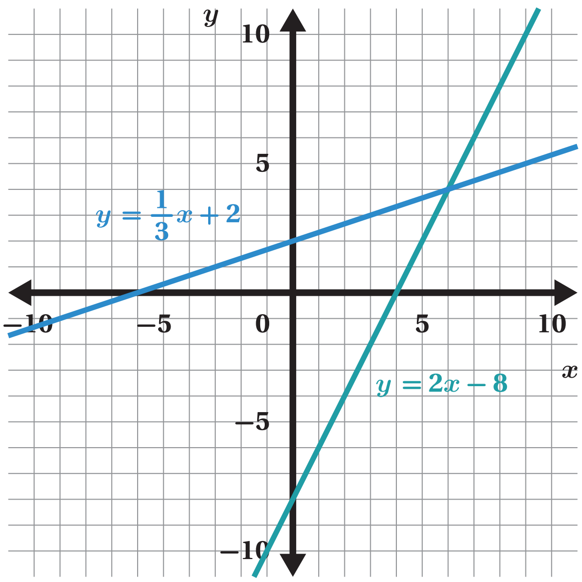
Summary

The four representations — table, graph, equation, and verbal description — of a linear relationship are all valuable when solving real-world problems.

Solutions to a problem can be represented as values in a table or points on a line. In an equation, the solutions are values that make the equation true.

Synthesis

Use the graph to complete the following problems.



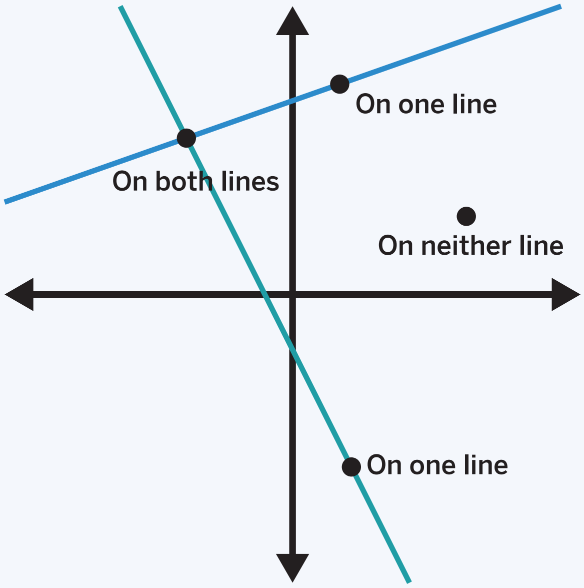
1. What is a combination of values that make both relationships true? How do you know?

2. What is a combination of values that make one relationship true, but not the other? How do you know?

Summary

Linear relationships can represent real-world scenarios. Two lines both graphed on the same coordinate plane can simultaneously represent the same scenario.

- The coordinates of a point that is on *both lines* make both relationships true. This is the intersection point.
- The coordinates of a point on only *one line* make only one relationship true.
- The coordinates of a point on *neither line* make neither relationship true.



Synthesis

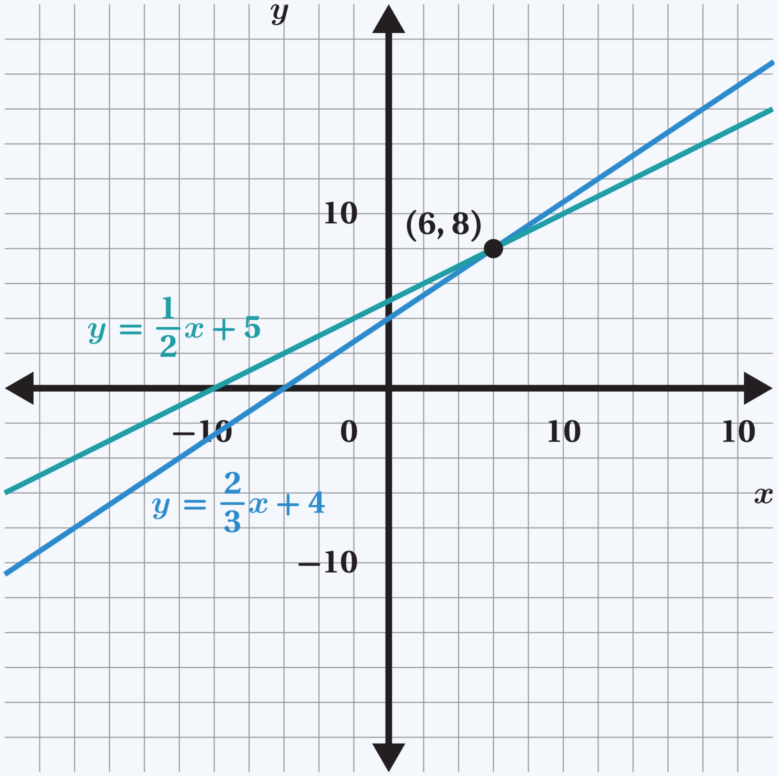
Describe something you would change about your poster now that you have seen other groups' work.

Summary

The solutions to an equation correspond to points on its graph. If there are two simultaneous equations, the solution can be found by studying the lines of the equations on the same coordinate plane and locating the coordinates of the intersection point. These coordinates represent the values of the two variables for which the two linear relationships will be the same.

For example, consider the graph shown.

When $x = 6$, both equations $y = \frac{1}{2}x + 5$ and $y = \frac{2}{3}x + 4$ have the same y-value of 8.



Synthesis

1. What is a system of equations?

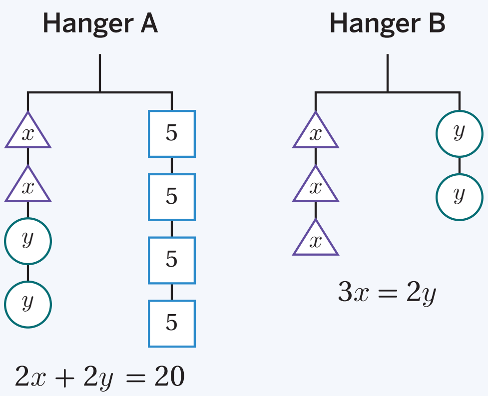
2. What does it mean to find the solution to a system of equations? Refer to the graph of the system of equations in the Summary if it helps with your thinking.

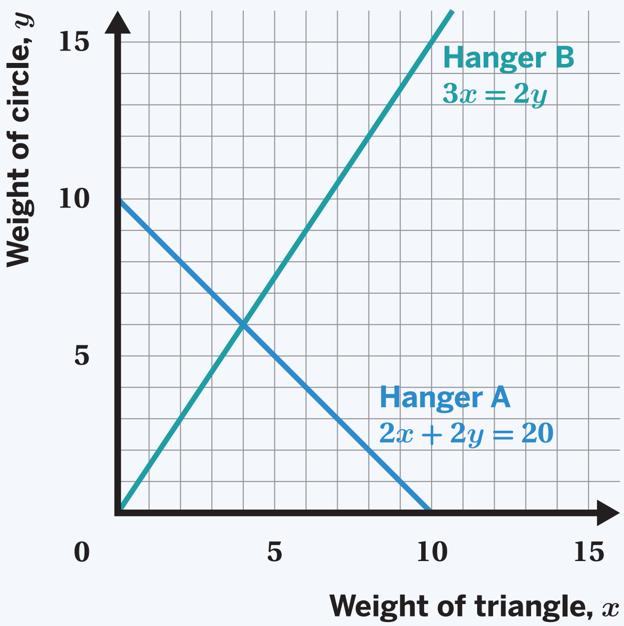
Summary

A **system of equations** is a set of two or more equations that share the same variables and are meant to be solved together.

A **solution to a system of equations** is an ordered pair that makes all equations in the system true.

For example, consider this system of equations: $\begin{cases} 2x + 2y = 20 \\ 3x = 2y \end{cases}$





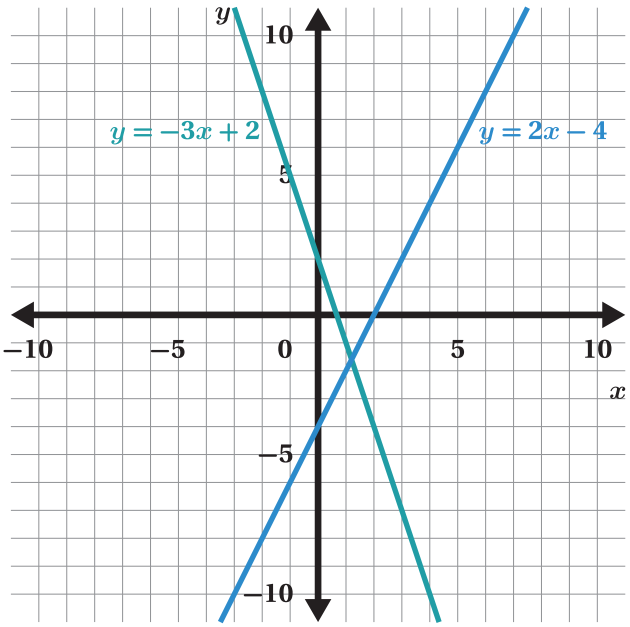
The lines intersect at the point $(4, 6)$, so the solution to the system of equations is the ordered pair $(4, 6)$. Both hangers will balance when the triangles weigh 4 and the circles weigh 6.

Synthesis

Here is the graph of this system of equations.

$$\begin{cases} y = -3x + 2 \\ y = 2x - 4 \end{cases}$$

How can you determine the *exact* solution to this system of equations?



Summary

For an ordered pair to be a solution to a system of equations, the x- and y-values of the ordered pair must make both of the equations true.

For example, consider this system of equations.

$$\begin{cases} y = 4x - 5 \\ y = -2x + 7 \end{cases}$$

To determine the solution to the system, write a single equation by taking the two expressions that are equal to y and setting them equal to each other.

$$4x - 5 = -2x + 7$$

$$6x - 5 = 7$$

$$6x = 12$$

$$x = 2$$

Then, substitute the solution for x into either of the original equations in the system to determine the value of y.

$$y = 4x - 5$$

$$y = 4(2) - 5$$

$$y = 8 - 5$$

$$y = 3$$

The solution to the system of equations is the ordered pair $(2, 3)$.

Synthesis

1. How can you tell from the *equations* the number of solutions a system of equations has?

2. How can you tell from the *graph* the number of solutions a system of equations has?

Summary

The number of solutions to a system of equations can be determined by looking at the graphs or by studying the coefficients and constants of the equations.

For example:

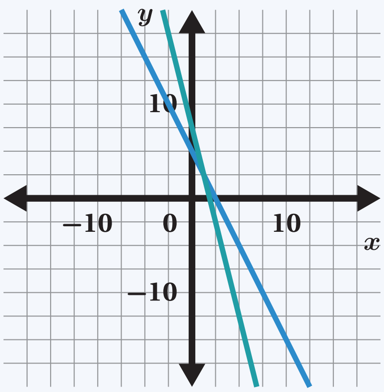
One solution:

$$\begin{cases} y = -4x + 8 \\ y = -2x + 5 \end{cases}$$

Equations:

- Different coefficients
- Same or different constants

Graph:



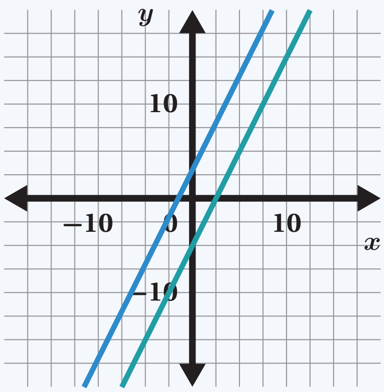
No solutions:

$$\begin{cases} y = 2x + 3 \\ y = 2x - 5 \end{cases}$$

Equations:

- Same coefficients
- Different constants

Graph:



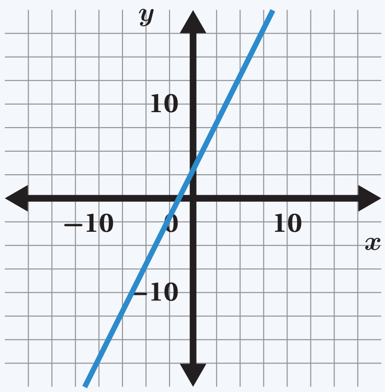
Infinitely many solutions:

$$\begin{cases} y = 2x + 3 \\ y = 2x + 3 \end{cases}$$

Equations:

- Same coefficients
- Same constants

Graph:



Synthesis

What is important to remember when solving a system of equations? Give an example system from the Partner Problems activity.

Summary

For an ordered pair to be a solution to a system of equations, the x- and y-values of the ordered pair must make both of the equations true. When one equation in the system is already solved for one variable — or can easily be put in this form — *substitution* is an efficient method for solving the system.

Consider this system of equations: $\begin{cases} y = 5x \\ 2x - y = 9 \end{cases}$

Because $y = 5x$, you can substitute $5x$ for y in $2x - y = 9$, and then solve for x.

$$2x - (5x) = 9$$
$$-3x = 9$$
$$x = -3$$

You can then substitute the solution for x into either of the original equations in the system to determine the value of y.

$$y = 5x$$
$$y = 5(-3)$$
$$y = -15$$

The ordered pair $(-3, -15)$ is the solution to the system of equations.