Unit 8

The Pythagorean Theorem and Irrational Numbers

There are lots of different types of triangles. What is the relationship between the sides of a right triangle? To answer this, a representation for the edge length of a square is needed. In this unit, you'll explore how side lengths of right triangles are related, and learn about new types of numbers.

Essential Questions

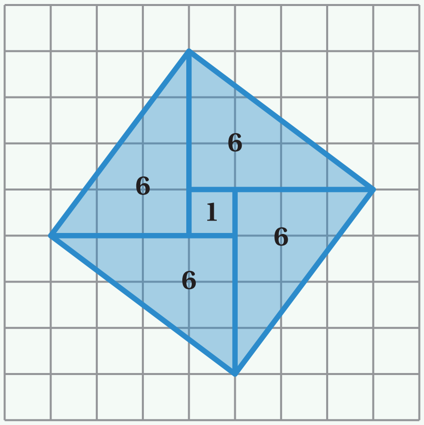
- How can you estimate the square root of a number? What does the square root of a number represent?
- Is it true that $\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2$ for all right triangles? If so, can you prove it?
- What is the difference between a rational number and an irrational number?

Summary | Lesson 1

There are many strategies for determining the area of a tilted square. Here are two strategies called "decompose and rearrange" and "surround and subtract."

Decompose and Rearrange

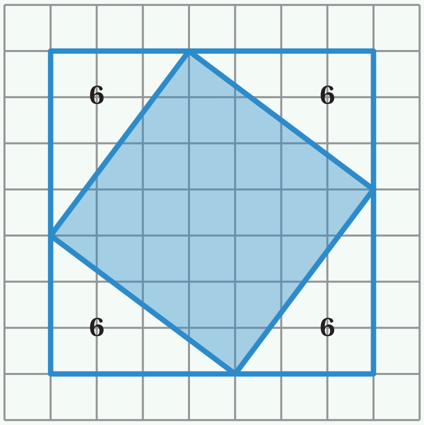
Area is calculated by adding the areas of the four triangles and one center square.



$4 \cdot 6 + 1 = 25$ square units

Surround and Subtract

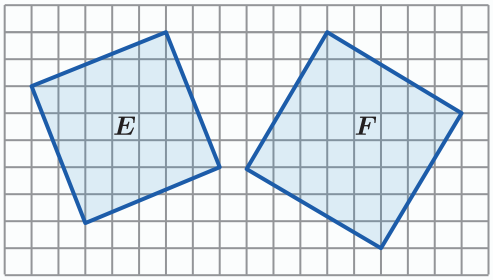
Area is calculated by finding the area of the large square minus the area of the four triangles.



$7 \cdot 7 - 4 \cdot 6 = 25$ square units

Try This

Use any strategy to calculate the area of each square.



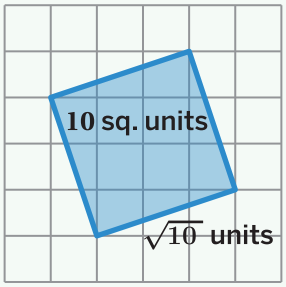
a Square E

b Square F

Summary | Lesson 2

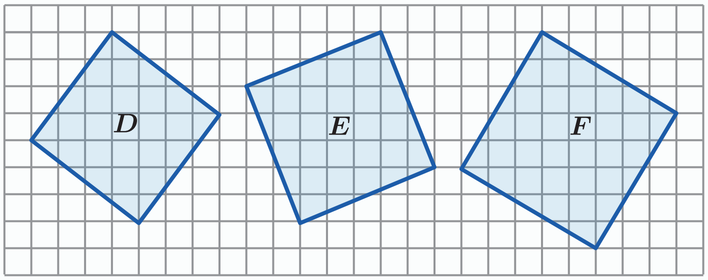
There is a known relationship between the area of any square and its side length. The exact value of the side length of a square can be written as the **square root** of its area.

For example, $\sqrt{10}$ is the exact value for the side length of a square with an area of 10 square units.



Try This

Complete the table with the missing values.



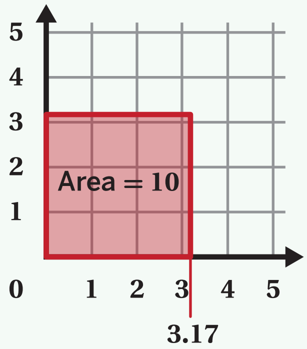
Square	Area (sq. units)	Side Length (sq. units)
D	25	
E		$\sqrt{29}$
F		

Summary | Lesson 3

You can use several strategies to approximate the values of square roots. One strategy is to use the areas of squares. The side length of a square is equal to the square root of its area. Another strategy is to create a table of values for n and determine n^2. Remember that $(\sqrt{n})^2 = n$. Below is a description of how each strategy can be used to approximate $\sqrt{10}$.

Using Squares

- Create a square that has an area equal to about 10 square units.
- Approximate the side length of the square created.



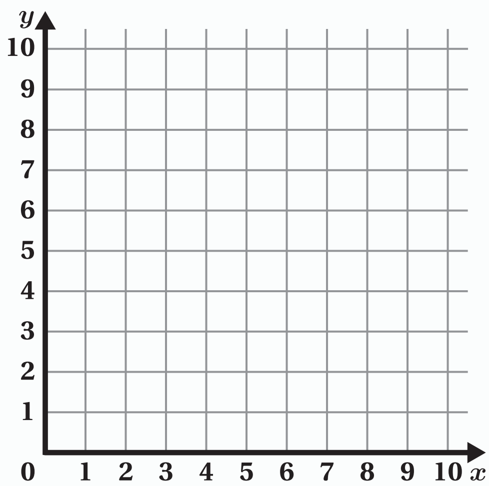
Using a Table

- Create a table of decimal value guesses for n.
- Calculate n^2 for each guess of n.
- The closer n^2 is to 10, the better that value of n is as an approximation for $\sqrt{10}$.

n	n^2
3.1	9.61
3.16	9.9856
3.17	10.0489
3.165	10.017225

Try This

Square B has an area of 17 square units. Estimate the side length of square B. Explain your thinking. Use the graph if it helps with your thinking.

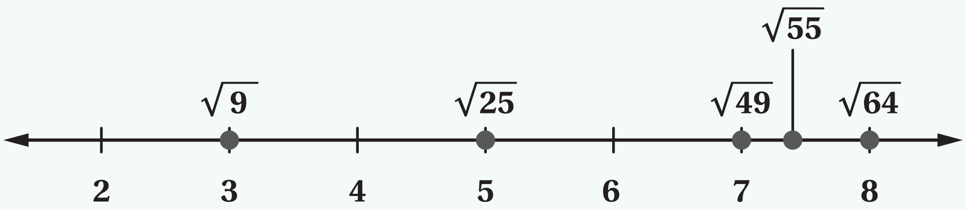


Summary | Lesson 4

You can represent a square root on a number line. We write a solution to an equation, such as $x^2 = 3$, using square root notation. The positive solution to this equation is $x = \sqrt{3}$.

You can approximate a square root on a number line by observing the whole numbers around it.

For example, you can determine that $\sqrt{55}$ is between 7 and 8 because $7^2 = 49$ and $8^2 = 64$, and 55 is between 49 and 64. More precisely, $\sqrt{55}$ should be plotted slightly left of 7.5 since it is closer to 7 than 8.



A **perfect square** is a number that is the square of an integer. For example, 49 is a perfect square because $7 \cdot 7 = 7^2 = 49$, but 55 is not a perfect square because no integer can be squared to equal 55.

Try This

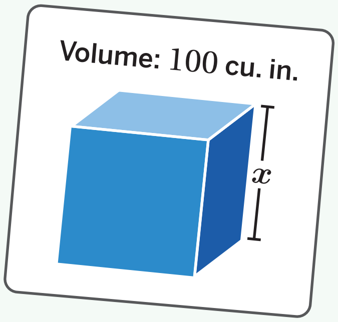
Sort each of the values into the correct category. Then plot each value on the number line.

$\sqrt{6}$	$\sqrt{12}$	$\sqrt{24}$	x when $x^2 = 8$

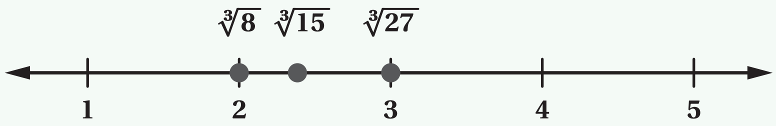
Between 2 and 3	Between 4 and 5

Summary | Lesson 5

A **cube root** describes the edge length of a cube given its volume. For the cube shown with a volume of 100 cubic inches, the equation $x^3 = 100$ can help you find its edge length. Its exact solution would be represented as $x = \sqrt[3]{100}$.



We can approximate a cube root on a number line by observing the whole numbers around it. For example, you can determine that $\sqrt[3]{15}$ is between 2 and 3 because $2^3 = 8$ and $3^3 = 27$, and 15 is between 8 and 27.



8 and 27 are perfect cubes because they are both the cube of an integer: $2 \cdot 2 \cdot 2 = 2^3 = 8$ and $3 \cdot 3 \cdot 3 = 3^3 = 27$.

Try This

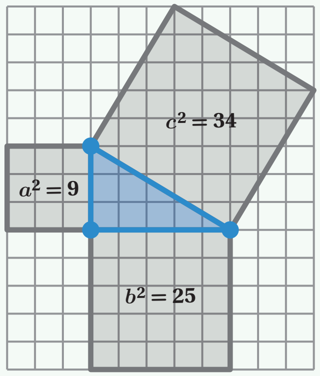
Complete the table without using a calculator.

Exact Cube Edge Length (units)	Approximate Cube Edge Length (units)	Volume of Cube (cu. units)
	Between ______ and ______	60
$\sqrt[3]{4}$	Between ______ and ______	
	Between ______ and ______	25

Summary | Lesson 6

The **Pythagorean theorem** says that for right triangles, $a^2 + b^2 = c^2$, where a and b represent the lengths of the two shorter sides and c represents the length of the longest side.

The relationship $a^2 + b^2 = c^2$ is only true for right triangles.



For triangle H:

$(\sqrt{10})^2 + (\sqrt{17})^2 = 27$

$c^2 = (\sqrt{29})^2 = 29$

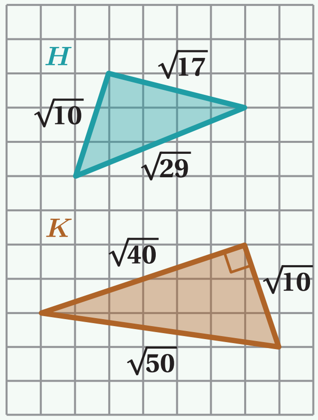
$27 \neq 29$ so $a^2 + b^2 = c^2$ is not true.

For triangle K:

$(\sqrt{10})^2 + (\sqrt{40})^2 = 50$

$c^2 = (\sqrt{50})^2 = 50$

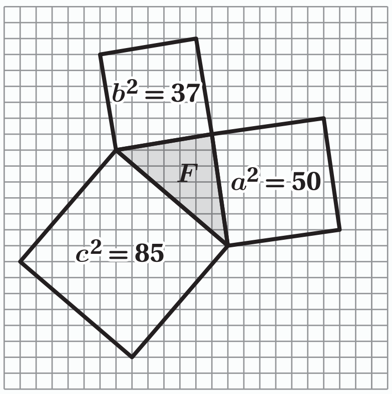
$50 = 50$ so $a^2 + b^2 = c^2$ is true.



Try This

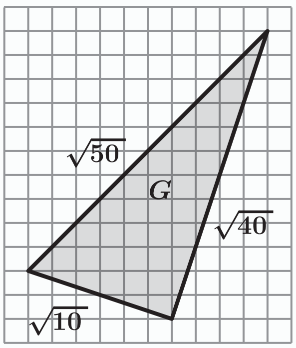
Use the Pythagorean theorem to determine if each triangle is a right triangle.

a



Is triangle F a right triangle? Explain your thinking.

b



Is triangle G a right triangle? Explain your thinking.

Summary | Lesson 7

The Pythagorean theorem states that for any right triangle, $a^2 + b^2 = c^2$. There are many proofs for the Pythagorean theorem.

One proof involves drawing squares on the sides of a right triangle, like in Figure 1. You can cut up pieces of the smaller squares to equal the area of the larger square.

Another involves rearranging triangles. Make a square where each side is $(a + b)$ units long (like Figures 2 and 3).

Start by placing 4 copies of the original right triangle into the square. In Figure 2, the remaining area is covered by two squares with total area $a^2 + b^2$. In Figure 3, the remaining area is covered by one square with area c^2. Since the area of the large square and the area of the 4 triangles are the same, $a^2 + b^2$ has to equal c^2.

Figure 1

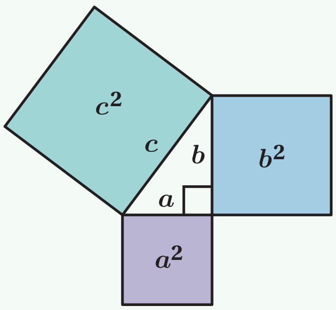


Figure 2

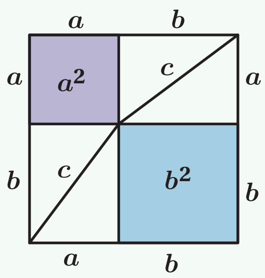
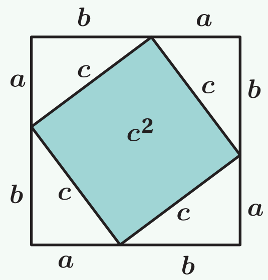
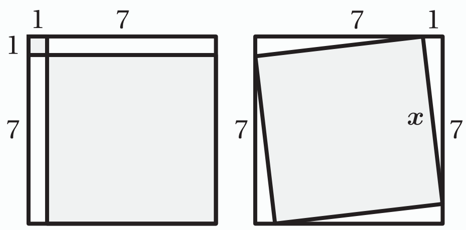


Figure 3

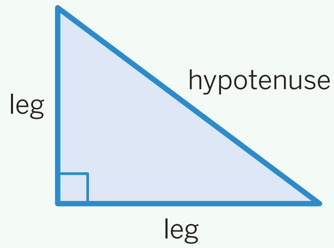


Try This

Determine the exact value of x.



Summary | Lesson 8



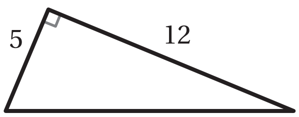
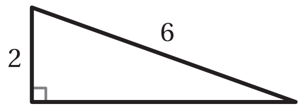
The **hypotenuse** is the side of a right triangle that is opposite the right angle, and is the longest side. Only right triangles have a *hypotenuse*. The **legs** of a right triangle are the sides that make the right angle.

The Pythagorean theorem says that in a right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse. This can be represented by the equation $a^2 + b^2 = c^2$, where a and b represent the lengths of the legs and c represents the length of the hypotenuse.

When any two side lengths of a right triangle are known, the Pythagorean theorem can be used to calculate the length of the third side, whether it is the hypotenuse or a leg. You can substitute the lengths you know into the equation $\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2$ or $a^2 + b^2 = c^2$, and then solve for the unknown value.

Try This

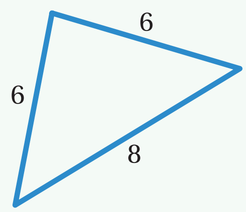
For each triangle in the table, determine whether the missing side is a leg or a hypotenuse. Then calculate the exact length of the missing side.

Triangle	Leg or Hypotenuse?	Missing Side Length
	Leg Hypotenuse	
	Leg Hypotenuse	

Summary | Lesson 9

If a triangle has side lengths a, b, and c, where c is the longest side and $a^2 + b^2 = c^2$, then the converse of the Pythagorean theorem says that you must have a right triangle. We can use this to determine whether any triangle is a right triangle. If the sides of a triangle *do not* make the equation $a^2 + b^2 = c^2$ true, then you know it is *not* a right triangle.

In the triangle shown, let $a = 6$, $b = 6$, and $c = 8$. You can use substitution to determine whether the triangle is a right triangle.

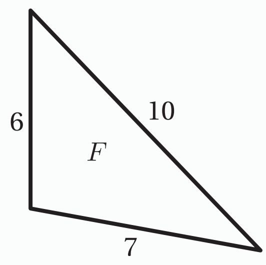


$$a^2 + b^2 = 36 + 36$$
$$= 72$$

Because $c^2 = 8^2$, or 64, the triangle cannot be a right triangle because $a^2 + b^2 \neq c^2$.

Try This

Change one of the side lengths to make triangle F a right triangle.

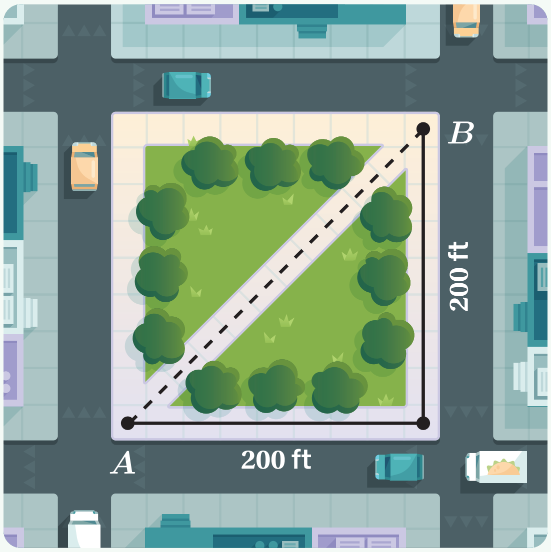


Summary | Lesson 10

The Pythagorean theorem can be used to solve problems that can be modeled with right triangles. The sides of a triangle might represent units such as the length of an object or the distance between two objects.

To apply the Pythagorean theorem, the lengths of two sides must be known so the length of the third side can be determined.

For example, you can use the Pythagorean theorem to calculate the distance to walk through the park from point A to point B.



Let the length of the path through the park equal c.

$$c^2 = (200)^2 + (200)^2$$

$$c^2 = 40000 + 40000$$

$$c^2 = 80000$$

$$c = \sqrt{80000}$$

Try This

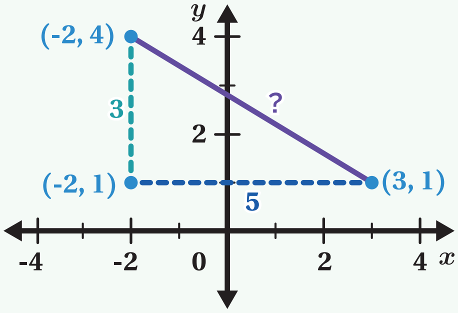
A 17-foot ladder is leaning against a wall. The ladder can reach a window 15 feet up the wall.

a Draw a picture of the situation.

b How far should the base of the ladder be from the wall so that it reaches the window? Show your thinking.

Summary | Lesson 11

You can use the Pythagorean theorem to calculate the distance between two points that are on a diagonal line segment. Start by drawing horizontal and vertical legs to form a right triangle. Then use the Pythagorean theorem to calculate the length of the hypotenuse, which will be the distance between the two points.



Calculate the distance between two points on a horizontal line by determining the absolute value of the difference between their x-coordinates. The distance between (-2, 1) and (3, 1) is $|-2 - 3| = 5$ units.

Similarly, you can calculate the distance between two points on a vertical line by determining the absolute value of the distance between their y-coordinates. The distance between points (-2, 4) and (-2, 1) is $|4 - 1| = 3$ units.

$$\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2$$

$$3^2 + 5^2 = c^2$$

$$9 + 25 = c^2$$

$$34 = c^2$$

$$\sqrt{34} = c$$

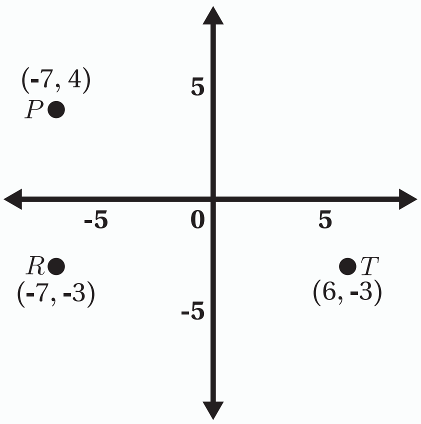
Try This

Calculate the length of each segment.

a Segment PR = ______ units

b Segment RT = ______ units

c Segment PT = ______ units



Summary | Lesson 12

You can write every number as a decimal. Some fractions can be written as terminating decimals, while others can be written as repeating decimals. To write a fraction as a decimal, you can use long division.

For example, here is how you can use long division to rewrite $\frac{1}{15}$ as a decimal.

$$\begin{array}{r} 0.06666... \\ 15\overline{)1.00} \\ \underline{-90} \\ 100 \\ \underline{-90} \\ 100 \\ \underline{-90} \\ 100 \\ \underline{-90} \\ 10 \end{array}$$

To avoid writing the repeating part of a decimal over and over, you can use **bar notation**, which shows a line over the part of the decimal that repeats. For example, when writing $\frac{1}{15}$ as a decimal, you would write $0.06666...$ as $0.0\overline{6}$.

Try This

a Sort the fractions based on whether they are terminating or repeating.

$\frac{3}{8}$ $\frac{3}{11}$ $\frac{98}{6}$

Terminating	Repeating

b Describe the strategy you used to determine whether each fraction represents a terminating or repeating decimal.

Summary | Lesson 13

You can express all repeating decimals as fractions. One way to do this is to multiply equations by factors of 10 until the repeating decimals can subtract to 0. Once the repetition is removed, the resulting equation can be solved and left in fraction form.

For example, see these steps to represent $0.\overline{57} = 0.575757575...$ as a fraction.

If a decimal expansion of a number is a repeating or terminating decimal, the number can be written as a fraction. If the digits in the decimal expansion do not repeat (non-repeating) and do not terminate (non-terminating), then the number cannot be written as a fraction.

$$\begin{aligned} x &= 0.\overline{57} \\ 10x &= 5.\overline{75} \\ 100x &= 57.\overline{57} \end{aligned}$$

$$\begin{aligned} 100x &= 57.\overline{57} \\ -(x &= 0.\overline{57}) \\ \hline 99x &= 57 \\ x &= \frac{57}{99} \\ 0.\overline{57} &= \frac{57}{99} \end{aligned}$$

Try This

a Match each repeating decimal to its equivalent fraction. One decimal does not have a match.

Repeating Decimal	**Fraction**
a. $5.\overline{37}$	.... $\frac{48}{9}$
b. $5.\overline{3}$	.... $\frac{34}{90}$
c. $0.\overline{53}$	.... $\frac{532}{99}$
d. $0.3\overline{7}$	

b Use any strategy to convert the remaining repeating decimal into a fraction.

Summary | Lesson 14

A **rational number** is a number that can be written as a fraction of two integers where the denominator is not zero. An **irrational number** is a number that cannot be written in that way.

Here are some examples of rational and irrational numbers.

Examples of Rational Numbers

- Fractions: $\frac{10}{5}$, $3\frac{11}{20} = \frac{71}{20}$
- Terminating decimals: $1.5 = \frac{3}{2}$, $1.73 = \frac{173}{100}$
- Repeating decimals: $1.\overline{73} = \frac{172}{99}$, $0.1212... = \frac{12}{99}$
- Square roots of perfect squares and cube roots of perfect cubes: $\sqrt[3]{8} = \frac{2}{1}$, $\sqrt{64} = \frac{8}{1}$, $\sqrt{\frac{1}{9}} = \frac{1}{3}$

Examples of Irrational Numbers

- Non-terminating, non-repeating decimals: π, $0.743...$, $2.742050...$
- Square roots of non-perfect squares and cube roots of non-perfect cubes: $\sqrt{2}$, $3 \cdot \sqrt{5}$, $\sqrt[3]{9}$

Try This

a Select *all* the irrational numbers.

- ☐ **A.** $\frac{5}{9}$
- ☐ **B.** $\sqrt{3}$
- ☐ **C.** $-\sqrt{25}$
- ☐ **D.** $0.2\overline{4}$
- ☐ **E.** $\sqrt[3]{25}$
- ☐ **F.** $-\pi$

b Remy says $\sqrt{27}$ is a rational number because $3^3 = 27$. Mio states that $\sqrt{27}$ is not rational because 27 is not a perfect square. Who do you agree with? Explain your thinking.

Try This | Answer Key

Lesson 1

a 29 square units

b 34 square units

Lesson 2

Square	Area (sq. units)	Side Length (sq. units)
D	25	**5**
E	**29**	$\sqrt{29}$
F	**34**	$\sqrt{34}$

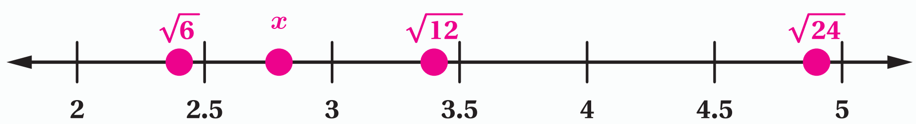
Lesson 3

Responses vary. Square B has an exact side length of $\sqrt{17}$, which lies between 4 and 5. Since $\sqrt{17}$ is closer to $\sqrt{16}$ than it is to $\sqrt{25}$, we can assume that $\sqrt{17}$ is closer to 4 than it is to 5. Using a table and calculator technology, $\sqrt{17} \approx 4.12$.

n	n^2
4	16
4.1	16.81
4.14	17.1396
4.12	16.9744
4.13	17.0569

Lesson 4

Between 2 and 3	Between 4 and 5
$\sqrt{6}$ x when $x^2 = 8$	$\sqrt{24}$



Lesson 5

Exact Cube Edge Length (units)	Approximate Cube Edge Length (units)	Volume of Cube (cu. units)
$\sqrt[3]{60}$	Between 3 and 4	60
$\sqrt[3]{4}$	Between 1 and 2	4
$\sqrt[3]{25}$	Between 2 and 3	25

Lesson 6

a No. *Explanations vary.* Because $37 + 50 \neq 85$, the Pythagorean theorem ($a^2 + b^2 = c^2$) is not true for this triangle.

b Yes. *Explanations vary.* The Pythagorean theorem is true for this triangle.

$$(\sqrt{10})^2 + (\sqrt{40})^2 = (\sqrt{50})^2$$

$$10 + 40 = 50$$

Lesson 7

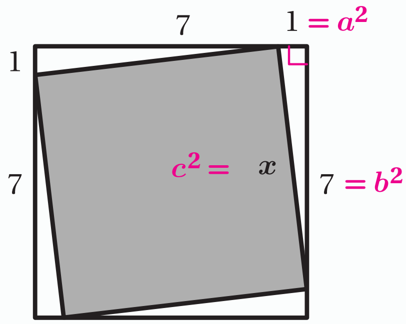
$$a^2 + b^2 = c^2$$

$$1^2 + 7^2 = x^2$$

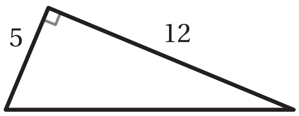
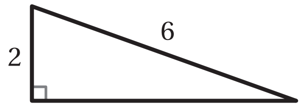
$$1 + 49 = x^2$$

$$50 = x^2$$

$$x = \sqrt{50}$$



Lesson 8

Triangle	Leg or Hypotenuse?	Missing Side Length
	Leg Hypotenuse	$5^2 + 12^2 = c^2$ $25 + 144 = c^2$ $169 = c^2$ $13 = c$
	Leg Hypotenuse	$2^2 + b^2 = 6^2$ $4 + b^2 = 36$ $-4 \qquad -4$ $b^2 = 32$ $b = \sqrt{32}$

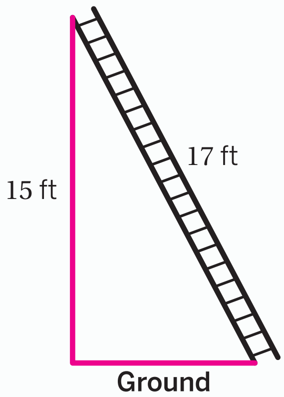
Lesson 9

Responses vary.

- Change 7 to 8, then $6^2 + 8^2 = 10^2$.
- Change 6 to $\sqrt{51}$, then $(\sqrt{51})^2 + 7^2 = 10^2$.
- Change 10 to $\sqrt{85}$, then $6^2 + 7^2 = (\sqrt{85})^2$.

Lesson 10

a



b 8 feet

$d^2 + 15^2 = 17^2$

$d^2 + 225 = 289$

$d^2 = 64$

$d = \sqrt{64}$

$d = 8$

Lesson 11

a. Segment $PR = 4 - (-3) = 7$ units

b. Segment $RT = 6 - (-7) = 13$ units

c. Segment $PT = \sqrt{7^2 + 13^2} = \sqrt{49 + 169} = \sqrt{218}$ units

Lesson 12

a.

Terminating	Repeating
$\frac{3}{8} = 0.375$	$\frac{3}{11} = 0.\overline{27}$ $\frac{98}{6} = 16.\overline{3}$

b. *Responses vary.* I used long division to determine whether each decimal was terminating or repeating. When I used long division to divide 3 into 8, there were no remainders left after a certain point, which makes the decimal representation of $\frac{3}{8}$ a terminating decimal. But when I divided 3 by 11 and when I divided 98 by 6, I eventually got to the point where I continued to get the same remainder, creating a pattern of repeating decimals.

$\frac{3}{8} = 0.375$ $\qquad$ $\frac{3}{11} = 0.\overline{27}$ $\qquad$ $\frac{98}{6} = 16.\overline{3}$

$$\begin{array}{r} 0.375 \\ 8\overline{)3.000} \\ -24 \\ \hline 60 \\ -56 \\ \hline 40 \\ -40 \\ \hline 0 \end{array} \qquad \begin{array}{r} 0.2727\ldots \\ 11\overline{)3.00000} \\ -22 \\ \hline 80 \\ -77 \\ \hline 30 \\ -22 \\ \hline 80 \\ -77 \\ \hline 30 \end{array} \qquad \begin{array}{r} 16.333\ldots \\ 6\overline{)98.00} \\ -6 \\ \hline 38 \\ -36 \\ \hline 20 \\ -18 \\ \hline 20 \\ -18 \\ \hline 20 \end{array}$$

Lesson 13

a

<u>b</u> $\frac{48}{9}$

<u>d</u> $\frac{34}{90}$

<u>a</u> $\frac{532}{99}$

b *Responses vary.*

$$\begin{aligned} x &= 0.\overline{53} \\ 100x &= 53.\overline{53} \\ 100x &= 53.\overline{53} \\ -(x &= 0.\overline{53}) \\ \hline 99x &= 53 \\ x &= \tfrac{53}{99} \text{ (or equivalent)} \end{aligned}$$

Lesson 14

a

B. $\sqrt{3}$

E. $\sqrt[3]{25}$

F. $-\pi$

b Mio. *Explanations vary*. 27 is a perfect cube and not a perfect square. If the number had been $\sqrt[3]{27}$, then Remy would be correct.